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Piecewise Polynomials and the Partition Method for Nonlinear Ordinary Differential Equations

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SUMMARY
An efficient numerical method, used previously for linear differential equations [1], is here extended to systems of nonlinear ordinary differential equations. Spline functions are used as the basic approximations. Residuals are liquidated by setting their integrals equal to zero over specified subintervals of the intervals of analyticity. Several diverse examples are given.

Notation

a	Radius of circular membrane
$[C'_j]$	An unsymmetrical 4×4 matrix, defined by Eq. (6)
$[C''_j]$	An unsymmetrical 6×6 matrix, defined by Eq. (8)
E	Young's modulus
F_i	Algebraic or transcendental functions
L	Differential operator
R	Closed interval
T	Dimensionless radius stress for the circular membrane
$[Y'_j]$	The transpose of the row matrix $[y_{j-1}, y'_{j-1}, y_j, y'_j]$
$[Y''_j]$	The transpose of the row matrix $[y_{j-1}, y'_{j-1}, y''_{j-1}, y_j, y'_j, y''_j]$
ε	Residual
ν	Poisson's ratio

1. Introduction

Problems involving one independent real variable x and one or more dependent real variables are considered. The range of x is a closed interval $R=[a, b]$.

Approximations introduce error terms, called residuals. For problems of ordinary differential equations, the residuals are functions of x and of certain parameters $(p_1, p_2, \ldots, p_s)$ whose values are to be chosen to liquidate the residuals in some sense. If ε is a residual, the simplest method is to set $\varepsilon=0$ at selected points in R. This is the collocation method. With the partition method [7], the interval R is divided into subintervals $R_1, R_2, \ldots, R_m$, and the following conditions are imposed:

$$\int_{R_j} \varepsilon(x, p_1, \ldots, p_s)dx = 0; \quad j=1, 2, \ldots, m\,. \tag{1}$$

If ε is not identically zero in R_j, Eq. (1) requires that it take both positive and negative values in R_j. Therefore, if ε is continuous, it vanishes at one or more points in R_j. Hence, the partition method satisfies the conditions of the collocation method automatically. Also, it balances the positive and negative values of ε against each other. As a result, the partition method is usually more accurate than the collocation method.

If the dependent variables are approximated by spline functions [2, 3, 4, 5], the interval R is divided into links in each of which the splines are analytic. For applications of the partition

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method, the intervals R_j are then conveniently taken to be certain subdivisions of these links.

The partition method, or any other parametric method, reduces the problem to a set of algebraic or transcendental equations,

$$F_i(p_1, p_2, \ldots, p_s) = 0; \quad i = 1, 2, \ldots, s . \tag{2}$$

If the differential-equation problem is nonlinear, some of the functions F_i are nonlinear.

A well-known method of successive extrapolations [10] has proved to be effective for solving Eq. (2). For short, the vector $(p_1, p_2, \ldots, p_s)$ is denoted as p. An initial estimate of the solution of Eq. (2) is p^0. By Taylor's theorem,

$$F_i(p) = F_i(p^0) + \sum_{k=1}^{s} (p_k - p_k^0)(\partial F_i/\partial p_k)_0 + \text{remainder} \tag{3}$$

where $(\partial F_i/\partial p_k)_0$ denotes the value of the designated derivative at p^0. Dropping the remainder, and setting $F_i(p) = 0$, for $i = 1, 2, \ldots, s$, we get linear equations that determine p_k. These values are used as the revised approximation p^0, and the process is repeated. The detailed discussion of convergence of this method can be found in reference [13]. In the subsequent examples, this method converges satisfactorily in less than nine iterations.

2. Construction of a Smooth Polynomial Chain

A finite closed interval $[a, b]$ of the x-axis is divided into n parts by points x_j, such that

$$a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b .$$

A chain of linear functions with prescribed ordinates y_j at points x_j is represented in matrix notation by

$$\tilde{y} = [1,\ x] \cdot [C_j] \cdot \begin{bmatrix} y_{j-1} \\ y_j \end{bmatrix} \tag{4}$$

where

$$[C_j] = (x_j - x_{j-1})^{-1} \begin{bmatrix} x_j & -x_{j-1} \\ -1 & 1 \end{bmatrix} .$$

More generally, a polynomial $y = f(x)$ of degree $2p+1$ is determined uniquely if the values of $y, y', y'', \ldots, y^{(p)}$ are given for $x = x_{j-1}$ and $x = x_j$, where primes denote derivatives and (x_{j-1}, x_j) are any distinct points. An interpolation formula of Fort [6] determined this polynomial. It is a generalization of Hermite interpolation [7]. If $p = 1$, the required polynomial is

$$\tilde{y} = [1,\ x,\ x^2,\ x^3] \cdot [C_j'] \cdot [Y_j'] \tag{5}$$

where $[1,\ x,\ x^2,\ x^3]$ is the indicated row matrix, $[Y_j']$ is the column matrix that is the transpose of the matrix $[y_{j-1},\ y'_{j-1},\ y_j,\ y'_j]$, and $[C_j']$ is an unsymmetrical 4×4 matrix, defined as follows:

$$\begin{aligned} &\text{Column } 1 = (x_j - x_{j-1})^{-3} [x_j^2 (x_j - 3x_{j-1}),\ 6x_{j-1}x_j,\ -3(x_{j-1} + x_j),\ 2] \\ &\text{Column } 2 = (x_j - x_{j-1})^{-2} [-x_{j-1}x_j^2,\ x_j(2x_{j-1} + x_j),\ -(x_{j-1} + 2x_j),\ 1] . \end{aligned} \tag{6}$$

Columns 3 and 4 are like Columns 1 and 2, respectively, except that x_j and x_{j-1} are interchanged throughout in Eq. (6). It is to be noted that the matrix $[C_j']$ is determined solely by x_j and x_{j-1}. When the partitioning of the interval $[a, b]$ is specified, the matrices $[C_j']$ may be computed. There is one $[C']$-matrix for each interval. Thus, a smooth chain of cubic polynomials is constructed.

If $p = 2$, the required polynomial is

$$\tilde{y} = [1,\ x,\ x^2,\ x^3,\ x^4,\ x^5] \cdot [C_j''] \cdot [Y_j''] \tag{7}$$

where $[Y_j'']$ is the transpose of the row matrix $[y_{j-1},\ y'_{j-1},\ y''_{j-1},\ y_j,\ y'_j,\ y''_j]$ and $[C_j'']$ is an unsymmetrical 6×6 matrix, defined by:

$$\text{Column } 1 = (x_j - x_{j-1})^{-5}[x_j^3(10x_{j-1}^2 - 5x_{j-1}x_j + x_j^2),\ -30x_{j-1}^2 x_j^2,\ 30x_{j-1}x_j(x_{j-1}+x_j),\ -10(x_{j-1}^2 + 4x_{j-1}x_j + x_j^2),\ 15(x_{j-1}+x_j),\ -6]$$

$$\text{Column } 2 = (x_j - x_{j-1})^{-4}[x_{j-1}x_j^3(4x_{j-1} - x_j),\ x_j^2(x_j + 2x_{j-1})(x_j - 6x_{j-1}),\ 6x_{j-1}x_j(2x_{j-1}+3x_j),\ -2(2x_{j-1}^2 + 10x_{j-1}x_j + 3x_j^2),\ 7x_{j-1} + 8x_j,\ -3]$$

$$\text{Column } 3 = \tfrac{1}{2}(x_j - x_{j-1})^{-3}[x_{j-1}^2 x_j^3,\ -x_{j-1}x_j^2(3x_{j-1}+2x_j),\ x_j(3x_{j-1}^2 + 6x_{j-1}x_j + x_j^2),\ -(x_{j-1}^2 + 6x_{j-1}x_j + 3x_j^2),\ 2x_{j-1} + 3x_j,\ -1]. \quad (8)$$

Columns 4, 5, 6 are like Columns 1, 2, 3, respectively, except that x_{j-1} and x_j are interchanged throughout in Eq. (8). Thus, a chain of fifth-degree polynomials is constructed with continuous first and second derivatives.

3. Nonlinear Ordinary Differential Equation

A second-order differential equation

$$L(y) = f(x) \quad (9)$$

is to be satisfied in a finite closed interval $[a, b]$, where L is a nonlinear (or linear) differential operator. Two boundary conditions are imposed. By hypothesis, there is a unique solution. The interval $[a, b]$ is partitioned into n parts as described previously. A piecewise cubic approximation of the solution is represented by Eq. (5); i.e.,

$$\tilde{y} = [1,\ x,\ x^2,\ x^3]\cdot[C_j']\cdot[Y_j'] \quad (10)$$

For convenience, let

$$[Y_j'] = \begin{bmatrix} \eta_{j,1} \\ \eta_{j,2} \\ \eta_{j,3} \\ \eta_{j,4} \end{bmatrix} \quad (11)$$

where

$$\eta_{j,1} = y_{j-1},\ \eta_{j,2} = y'_{j-1},\ \eta_{j,3} = y_j \text{ and } \eta_{j,4} = y'_j.$$

Then

$$\tilde{y} = g_{j,1}\eta_{j,1} + g_{j,2}\eta_{j,2} + g_{j,3}\eta_{j,3} + g_{j,4}\eta_{j,4}$$

or

$$\tilde{y} = \sum_{i=1}^{4} g_{j,i}\eta_{j,i}, \qquad (x_{j-1} \leqq x \leqq x_j) \quad (12)$$

where

$$g_{j,i} = (C_j')_{1i} + (C_j')_{2i}x + (C_j')_{3i}x^2 + (C_j')_{4i}x^3 \qquad i = 1, 2, 3, 4. \quad (13)$$

Substituting Eq. (12) into Eq. (9) we have

$$L(\tilde{y}) = L\left(\sum_{i=1}^{4} g_{j,i}\eta_{j,i}\right) = f(x) + \varepsilon(x) \qquad (x_{j-1} \leqq x \leqq x_j) \quad (14)$$

where $\varepsilon(x)$ is the residual resulting from the approximation. If $\varepsilon(x)$ were identically zero, $\tilde{y}$ would be the exact solution. Although this condition is generally unattainable, it is possible to set $\varepsilon(x) = 0$ at $2n$ points in the interval $[a, b]$. Then, with the boundary conditions, $2n+2$ nonlinear algebraic equations are obtained for the unknowns (y_j, y_j'); $j = 0, 1, \ldots, n$. This is the collocation method.

A more accurate procedure that often is as easy to apply as the collocation method is the partition method [7]. For a second-order differential equation each interval $[x_{j-1}, x_j]$ is divided into two parts by a point ξ_j, chosen so that $x_{j-1} < \xi_j < x_j$. Then the definite integral

$\int \varepsilon(x)dx$ is set equal to zero for each subdivision. Thus, $2n$ equations are obtained. Then, with the boundary conditions, there are enough equations to determine all the unknowns (y_j, y'_j). In some cases, a minor difficulty arises because the integrals cannot be evaluated in closed form. However, numerical integration can be used.

The foregoing method applies also to systems of nonlinear (or linear) differential equations. For instance, three such differential equations are represented by

$$L_{r1}(u)+L_{r2}(v)+L_{r3}(w)=f_r(x)\ ; \qquad r=1,2,3 \tag{15}$$

where L_{rs} is a second-order nonlinear (or linear) differential operator. Again, the interval $[a, b]$ is divided into n parts by points x_j, and each part is divided in two by a point ξ_j. The functions u, v, w are approximated by

$$\begin{aligned} \tilde{u} &= [1,\ x,\ x^2,\ x^3]\cdot[C'_j]\cdot[U'_j] \\ \tilde{v} &= [1,\ x,\ x^2,\ x^3]\cdot[C'_j]\cdot[V'_j] \\ \tilde{w} &= [1,\ x,\ x^2,\ x^3]\cdot[C'_j]\cdot[W'_j]\,. \end{aligned} \tag{16}$$

Here $[U'_j]$ is the column matrix $[u_{j-1},\ u'_{j-1},\ u_j,\ u'_j]$, etc. Equations (16) are substituted into Eq. (15), and the integrals of the residuals over each part are set equal to zero, as before. Thus $6n$ nonlinear (or linear) algebraic equations are obtained, and with the six boundary conditions, there are enough equations to determine all the unknowns.

A third-order differential equation, $L_1(y)=f(x)$, can be reduced to one first-order and one second-order differential equation by the substitution $y'=u$. A fourth-order differential equations $L_2(y)=f(x)$, can be reduced to two second-order differential equations by letting $y''=u$. The piecewise cubic approximation consequently serves for third and fourth order differential equations.

Innumerable variations of the preceding method can be devised. The methods of Ritz, Galerkin, least squares, or orthogonality [7] might be used instead of collocation or partition, or several of these methods might be combined. By suitable substitutions, any system of differential equations can be reduced to the first order. Then piecewise linear approximations can be used Eq. (4). On the other hand, equations of the third or fourth order might be handled conveniently by piecewise quintic approximations, in accordance with Eqs. (7) and (8). An obvious difficulty occurs if the interval $[a, b]$ is infinite, but, if the solution attenuates the infinite interval can be replaced by a finite one.

An advantage of the present method is that it yields directly the first derivatives in the case of a piecewise cubic approximation, or the first and second derivatives in the case of a piecewise quintic approximation, in addition to the value of the function. In physical problems, the values of the derivatives are often required.

Example 1. *Second-Order Nonlinear Differential Equation* (Boundary Value Problem)

$$L(y)=yy''+y'^2-a^2=0;\quad y(0)=1,\ y(1)=2\,. \tag{17}$$

By Eq. (14)

$$L(\tilde{y})=\left(\sum_{i=1}^{4} g_{j,i}\eta_{j,i}\right)\left(\sum_{i=1}^{4} g''_{j,i}\eta_{j,i}\right)+\left(\sum_{i=1}^{4} g'_{j,i}\eta_{j,i}\right)^2-a^2=\varepsilon(x), \qquad (x_{j-1}\leqq x\leqq x_j) \tag{18}$$

Integration yields

$$\int_j L(\tilde{y})dx = \int_j \varepsilon(x)dx \tag{19}$$

or

$$\begin{aligned}\int_j (g_{j,1}g''_{j,1}+g'^2_{j,1})dx\,\eta^2_{j,1}&+\int_j (g_{j,1}g''_{j,2}+g_{j,2}g''_{j,1}+2g'_{j,1}g'_{j,2})dx\,\eta_{j,1}\eta_{j,2}\\ &+\int_j (g_{j,1}g''_{j,3}+g_{j,3}g''_{j,1}+2g'_{j,1}g'_{j,3})dx\,\eta_{j,1}\eta_{j,3}\\ &+\int_j (g_{j,1}g''_{j,4}+g_{j,4}g''_{j,1}+2g'_{j,4}g'_{j,1})dx\,\eta_{j,1}\eta_{j,4}\\ &+\int_j (g_{j,2}g''_{j,2}+g'^2_{j,2})dx\,\eta^2_{j,2}\\ &+\int_j (g_{j,2}g''_{j,3}+g_{j,3}g''_{j,2}+2g'_{j,2}g'_{j,3})dx\,\eta_{j,2}\eta_{j,3}\\ &+\int_j (g_{j,2}g''_{j,4}+g_{j,4}g''_{j,2}+2g'_{j,2}g'_{j,4})dx\,\eta_{j,2}\eta_{j,4}\\ &+\int_j (g_{j,3}g''_{j,3}+g'^2_{j,3})dx\,\eta^2_{j,3}\\ &+\int_j (g_{j,3}g''_{j,4}+g_{j,4}g''_{j,3}+2g'_{j,3}g'_{j,4})dx\,\eta_{j,3}\eta_{j,4}\\ &+\int_j (g_{j,4}g''_{j,4}+g'^2_{j,4})dx\,\eta^2_{j,4}-a^2\int_j dx = \int_j \varepsilon(x)dx\,. \end{aligned} \tag{20}$$

The interval $[0, 1]$ is divided into n parts by points, $0=x_0<x_1<\ldots<x_n=1$ and an intermediate point ξ_j is inserted in each part.

Points ξ_i were chosen as the midpoints of the intervals i.e., for $n=2$ $\xi_1=\frac{1}{4}$ and $\xi_2=\frac{3}{4}$. Set the integral of the residual equal to zero in each subinterval. Then four equations are generated from Eq. (19). Two more equations are given by the boundary conditions $y(0)=1$ and $y(1)=2$; this implies $\eta_{1,1}=1$ and $\eta_{2,3}=2$. The six unknowns, η_1, η_2, η_3, η_4, η_5 and η_6 can be determined from the above 6 nonlinear algebraic equations by the iteration method. A computer program was written to solve the problem for various values of n with intervals of equal (or unequal) length. If the maximum differences of absolute values of solutions of two successive iterations is less than (.00001), the solution is considered to be suitable. Table 1 shows the results for different n. The convergence of the iteration process is very fast; hence it does not use much computer time. For this example all initial estimates were taken to be 0.5, and the allowable maximum difference for two consecutive iterations was taken as small as 0.00001. At most seven iterations were required to get final results for different n. Some of the examples which are given later need

TABLE 1

Solution of eq. (17) *for* $a=2$
Exact solution is $y=\sqrt{4x^2-x+1}$

x_j			0.00	0.20	0.25	0.35	0.50	0.70	0.75	0.85	1.00
Exact		y_j	1.00000	0.97980	1.00000	1.06771	1.22474	1.50333	1.58114	1.74356	2.00000
		y'_j	−0.50000	0.30619	0.50000	0.84293	1.22474	1.52994	1.58114	1.66326	1.75000
Partition and iteration method	$n=2$	y_j	1.00000	—	—	—	1.22526	—	—	—	2.00000
		y'_j	−0.50197	—	—	—	1.22262	—	—	—	1.74902
	$n=4$	y_j	1.0000	—	0.99998	—	1.22476	—	1.58115	—	2.00000
		y'_j	−0.50009	—	0.49992	—	1.22465	—	1.58107	—	1.74995
	$n=6$	y_j	1.00000	0.97979	—	1.06770	1.22474	1.50333	—	1.74356	2.00000
		y'_j	−0.50000	0.30617	—	0.84292	1.22474	1.52993	—	1.66326	1.74999

TABLE 2

Comparison between exact solution and solution given by partition method for example 2
Exact solution is $x^3 y^2 = x^2 + 2x + 1$

x_j		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Exact	y_j	2.00000	1.82025	1.67360	1.55172	1.44884	1.36083	1.28468	1.21812	1.15944	1.10731	1.06066
Partition method	y_j	2.00000	1.82025	1.67360	1.55172	1.44884	1.36083	1.28467	1.21812	1.15944	1.10730	1.06066
Error		0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00001	0.00000
Exact	y'_j	−2.00000	−1.61537	−1.33127	−1.11578	−0.94864	−0.81650	−0.71028	−0.62366	−0.55212	−0.49236	−0.44194
Partition method	y'_j	−2.00000	−1.61537	−1.33127	−1.11579	−0.94865	−0.81650	−0.71028	−0.62366	−0.55212	−0.49236	−0.44195
Error		0.00000	0.00000	0.00000	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
Exact	y''_j	4.50000	3.27730	2.45651	1.88625	1.47840	1.17938	0.95540	0.78445	0.65180	0.54736	0.46404
Partition method	y''_j	4.49994	3.27725	2.45647	1.88622	1.47838	1.17937	0.95539	0.78444	0.65180	0.54736	0.46404
Error		0.00006	0.00005	0.00004	0.00003	0.00002	0.00001	0.00001	0.00001	0.00000	0.00000	0.00000

only three or four iterations. In general, most problems do not exceed ten iterations, even with a large number of unknowns and a poor initial estimate.

Example 2. *Third-Order Nonlinear Differential Equation* (Boundary Value Problem)

$$L(y) = x^3 y y''' + 3x^3 y' y'' + 9x^2 y y'' + 9x^2 y'^2 + 18xyy' + 3y^2 = 0 \quad \text{(a)}$$

and

$$y(1) = 2, \quad y'(1) = -2 \quad \text{and} \quad y''(2) = 0.46404. \quad \text{(b)}$$

The general solution of Eq. (a) is

$$x^3 y^2 = C_1 x^2 + C_2 x + C_3$$

The piecewise quintic polynomial approximation is well adapted to this problem. However, piecewise cubic polynomials will be used in order to show that lower order spline functions can be used to approximate the solutions of higher order nonlinear differential equations, and to illustrate how the partition method can be employed to solve a system of simultaneous differential equations.

The substitution $z = y'$ reduces Eq. (a) from third order to second order. Then, the differential equations which are to be solved are

$$x^3 yz'' + 3x^3 zz' + 9x^2 yz' + 9x^2 z^2 + 18xyz + 3y^2 = 0 \quad \text{(c)}$$
$$y' - z = 0$$

The boundary conditions (b) become

$$\begin{aligned} y &= 2 && \text{at } x = 1 \\ z &= -2 && \text{at } x = 1 \\ z' &= 0.46404 && \text{at } x = 2 \end{aligned} \quad \text{(d)}$$

The interval [1, 2] is divided into ten parts, and intermediate points ξ_j are inserted at the centers of the parts. Since Eqs. (c) contain two dependent variables, y and z, the approximations are

$$\tilde{y} = [1,\ x,\ x^2,\ x^3]\ [C_j']\ [Y_j']$$
$$\tilde{z} = [1,\ x,\ x^2,\ x^3]\ [C_j']\ [Z_j'] \quad \text{(e)}$$

Equations (e) are substituted into Eqs. (c), and the integrals of the residuals over each subdivision are set equal to zero. Thus 40 nonlinear algebraic equations are obtained for (y_j, y_j', z_j, z_j'), $j = 0, 1, 2, \ldots, 10$ by applying Eq. (19). Three more equations result from the boundary conditions. One more equation can be obtained by setting $\varepsilon(x) = 0$ at any one partition point or by setting the sum of the integrals of the residuals is equal to zero over a subinterval which is different from the previous ones. For the computations, we arbitrarily set $H = (x_1 - x_0)/3$ and selected the subinterval $(x_0 + H,\ x_0 + 2H)$. The 44 unknowns can be determined from the 44 nonlinear algebraic equations by using the iteration process described in the Introduction. The numerical solution and exact solution are given in Table 2. Table 2 shows that y_j and y_j' are almost identical with the exact solution up to five decimal places, and the maximum error of y_j'' is 0.00006 at $x = 1$.

If $z = y''$ were introduced instead of $z = y'$, the problem would again be solvable. Thus, the third derivatives y''' could be obtained at the junctions of the parts.

Example 3. *Nonlinear Boundary Value Problem for the Circular Membrane*

A circular membrane of radius a and thickness t is subjected to a normal pressure $p(r)$, where r is the radial coordinate. If the deformation is axisymmetric, the radial and circumferential membrane stresses σ_r and σ_θ and the radial and normal displacements U and W are functions of r only, the circumferential displacement and the shear stress vanish. Then Föppl's membrane equations [8] can be reduced to the single nonlinear ordinary differential equation for the dimensionless radial stress σ, [9]

$$L(\sigma)=(x^3\sigma')'+\frac{Q}{\sigma^2}=0 \tag{a}$$

where $x=r/a$, $0\leqq x\leqq 1$, $\sigma(x)=\sigma_r(r)/E$

$$Q(x)=\left(\frac{Q_0}{x}\right)\left[\int_0^{ax}(p(\zeta)/E)\zeta\,d\zeta\right]^2\geqq 0 \tag{b}$$

$$Q_0=\frac{1}{2t^2a^2}$$

and E is Young's modulus.

We consider the case $p(r)=$ constant. Then from Eq. (b)

$$Q(x)=\tfrac{1}{8}\left(\frac{pa}{Et}\right)^2x^3. \tag{c}$$

A new dependent variable T is defined by

$$T(x)=\frac{2\,\sigma(x)}{\left(\frac{pa}{Et}\right)^{\frac{2}{3}}}. \tag{d}$$

Then the boundary value problem is reduced to

$$L(T)=(x^3T')'+x^3/T^2=0;\quad T'(0)=T'(1)+(1-\nu)T(1)=0 \tag{e}$$

For the numerical computations, Poisson's ratio ν was set equal to 0.32. The interval (0, 1) was partitioned into 5 equal parts, and the points ξ_j were chosen as the centers of the parts. Equations (19) and their counterparts for the intervals (ξ_j, x_j) provide 10 equations for T_j and T'_j. The boundary conditions

$$T'(0)=T'(1)+(1-\nu)\,T(1)=0$$

give two more equations. Thus T_j and T'_j were determined at the net points $x=0$, 0.2, 0.4, 0.6, 0.8 and 1.0. The numerical solution is given in the following table.

x_j	0.0	0.2	0.4	0.6	0.8	1.0
T_j	0.8676	0.8613	0.8385	0.8008	0.7462	0.6695
T'_j	0.0000	−0.0890	−0.1591	−0.2333	−0.3262	−0.4553

Since the differential equation (e) for the circular membrane is singular, Keller's theorem [11] does not apply. The shooting method was applied to obtain a numerical solution of the fixed-edge membrane with uniform normal pressure by L. Bauer [9]. The largest value of the dimensionless radial stress T is 0.8676 at $x=0$. The comparison of the present results and those of the shooting method are shown in Figure 1.

Example 4. Initial-Value Problem

If a second-order differential equation is to be solved with initial values (y_1, y'_1), the vanishing of the integral of the residual of the cubic approximation over the two subdivisions of the first interval provides two equations which determine (y_2, y'_2). Then the same process, applied to the second interval, determines (y_3, y'_3) and so on.

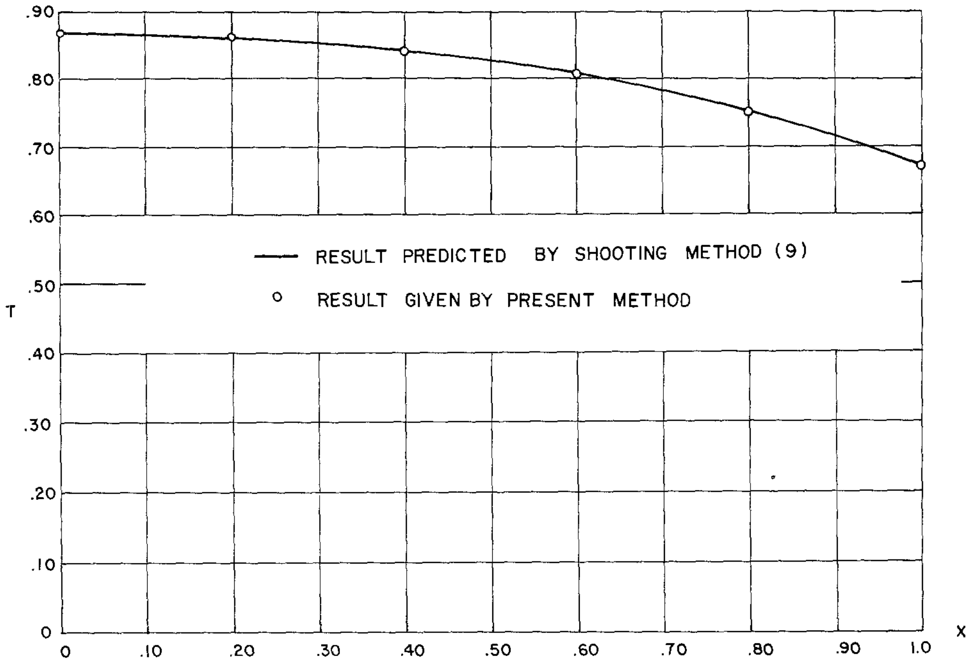

Figure 1. Variation of the dimensionless radial stress T. $\nu=0.32$.

As an example, the following problem is considered:

$$L(y)=xy''+\frac{x}{y}y'^2-3y'=0;\quad y(1)=1,\ y'(1)=1\,. \tag{a}$$

With $x_1=1$, $y_1=1$, $y_1'=1$, $x_2=1.5$, $\xi=1.25$, Eq. (19) and its counterpart for the segment (1.25, 1.5) yield two equations for (y_2, y_2'). The solution is $y_2=1.74109$, $y_2'=1.93855$. Then, with these starting values and $x_2=1.5$, $\xi=1.75$, $x_3=2$, a similar calculation gives $y_3=2.91548$, $y_3'=2.74398$. Likewise, stepping from $x_3=2$ to $x_4=2.5$, and then from x_4 to $x_5=3$, and so on, we get $y_4=4.47574$, $y_4'=3.49122$, $y_5=6.40328$, $y_5'=4.21680$.

In Table 3, the values corresponding to $x=1, 2, 3, 4$ were computed by the preceding method, with steps of unit length. Then the intermediate values were obtained by solving three boundary-value problems. For example, the values at $x=2.2, 2.4, 2.6, 2.8$ were computed by using a piecewise cubic approximation with five links in the interval [2, 3], and adapting it to the boundary conditions, $y(2)=2.91551$, $y(3)=6.40320$ which are given in Table 3. The partition method was again applied to Eq. (a) to solve this problem. This method of interpolation appears to be preferable to progressive projections over short steps.

Example 5. Van Der Pol's Equation

According to the mode of production of the non-linear variable damping, there exist two principal types of non-linear self-excited oscillations governed by (1) Van der Pol's equation and (2) Rayleigh's equation. In this example we solve Van der Pol's equation only; the same procedure can be applied to the other. The classical nonlinear differential equation of Van der Pol can be written as

$$\ddot{x}-\mu(1-x^2)\dot{x}+x=0 \tag{a}$$

where μ is a damping factor. This is the equation for an oscillatory system having variable damping. If the displacement x is small, the coefficient of $\dot{x}$ is negative and the damping is negative. If the displacement is large, damping becomes positive. The qualitative nature of the solution depends on the value of the parameter μ. We consider the initial values,

TABLE 3

Comparison between exact solution and solution given by partition method for example 4

First step	x_j		1.0	1.2	1.4	1.6	1.8	2.0
	Partition method	y_j	1.00000	1.23969	1.55591	1.94342	2.39769	2.91551
		y'_j	1.00000	1.39393	1.76364	2.10768	2.43240	2.74401
	Exact solution	y_j	1.00000	1.23968	1.55589	1.94340	2.39767	2.91548
		y'_j	1.00000	1.39391	1.76362	2.10765	2.43237	2.74398
	x_j		2.0	2.2	2.4	2.6	2.8	3.0
Second step	Partition method	y_j	2.91551	3.49472	4.13391	4.83212	5.58870	6.40320
		y'_j	2.74401	3.04695	3.34413	3.63741	3.92802	4.21674
	Exact solution	y_j	2.91548	3.49468	4.13386	4.83206	5.58863	6.40312
		y'_j	2.74398	3.04692	3.34409	3.63737	3.92797	4.21669
	x_j		3.0	3.2	3.4	3.6	3.8	4.0
Third step	Partition method	y_j	6.40320	7.27530	8.20478	9.19145	10.23519	11.33591
		y'_j	4.21674	4.50410	4.79049	5.07613	5.36123	5.64589
	Exact solution	y_j	6.40312	7.27522	8.20468	9.19134	10.23508	11.33578
		y'_j	4.21669	4.50406	4.79044	5.07608	5.36117	5.64584

TABLE 4

Solution of Van Der Pol's equation with $x(0)=0$, $\dot{x}(0)=2$ *and* $\mu=0.05$.

First step	t_j	0	0.1	0.2	0.3	0.4	0.5
	x_j	0.00000	0.20016	0.39928	0.59523	0.78585	0.96898
	$\dot{x}_j$	2.00000	1.99987	1.97891	1.93649	1.87234	1.78661
Second step	t_j	0.5	0.6	0.7	0.8	0.9	1.0
	x_j	0.96898	1.14247	1.30430	1.45256	1.58553	1.70171
	$\dot{x}_j$	1.78661	1.67993	1.55345	1.40880	1.24806	1.07366
Third step	t_j	1.0	1.1	1.2	1.3	1.4	1.5
	x_j	1.70171	1.79989	1.87910	1.93866	1.97817	1.99749
	$\dot{x}_j$	1.07366	0.88829	0.69473	0.49581	0.29419	0.09239

$$x(0)=0,\ \dot{x}(0)=2,\ \text{and}\ \mu=0.05\,.$$

The procedure of Example 4 was applied to this problem. The numerical solution is given in Table 4. No simple analytical solution of van der Pol's equation is known. However, the first order approximate solution [12] was employed for comparison. The difference of the two solutions is less than 4 percent.

If we set $\mu=0$ in the same computer program, the solution can be compared to the exact solution, since then the Eq. (a) is linear. The maximum error is not greater than 0.00001.

Example 6. Second-Order Differential Equation with Singular Point

$$L(y)=xyy''+xy'^2+kyy'=0 \tag{a}$$

$$y(1)=0 \text{ and } y'(1)=0\,. \tag{b}$$

If $x=0$, Eq. (a) reduces to

$$yy'=0 \tag{c}$$

TABLE 5

Comparison between exact solution and solution given by partition method for example 6.

x_j		1.000	0.800	0.600	0.400	0.300	0.200
Partition method	y_j	1.00000	0.83952	0.75153	0.71610	0.70997	0.70767
	y'_j	1.00000	0.60987	0.28741	0.08938	0.03804	0.01133
Exact solution	y_j	1.00000	0.83952	0.75153	0.71610	0.70996	0.70767
	y'_j	1.00000	0.60987	0.28741	0.08937	0.03803	0.01130

x_j		0.100	0.008	0.006	0.004	0.002	0.001
Partition method	y_j	0.70711	0.70711	0.70711	0.70711	0.70711	0.70711
	y'_j	0.00010	0.00005	0.00002	0.00001	0.00000	0.00000
Exact solution	y_j	0.70711	0.70711	0.70711	0.70711	0.70711	0.70711
	y'_j	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Therefore if either y or y' is zero, the other can be any arbitrary value. Hence $x=0$ is a singular point [10]. For the numerical computations of y and y' near $x=0$, k was set equal to -3. The interval (0, 1) was partitioned so that points were crowded near the origin. The values of y_j, y'_j and the exact solution are given in Table 5. By extrapolation, $y(0)=0.70711$ and $y'(0)=0$. These agree with the correct values.

Conclusions

Piecewise cubic and piecewise quintic polynomials may be used in conjunction with the partition method and the iteration method to provide approximate solutions of boundary-value problems and initial-value problems of nonlinear and linear differential equations. An advantage of the present method is that it yields directly the first order derivatives in the case of a piecewise cubic approximation, or the first and second derivatives in the case of a piecewise quintic approximation, as well as the function values at the net points. The values of the derivatives are sought in many physical problems. This method is allied closely to the popular finite-element method [2], but it is independent of variational formulations and physical principles. The convergence of the iteration process is generally fast: hence the computing time is not excessive. The examples that have been treated indicate that the method is comparatively reliable, accurate, and adaptable. The interval may be partitioned with unequal lengths so that the points are densest in intervals where high gradients are anticipated.

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